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A NOTE ON GROUPS WITH PROJECTIONS

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The purpose of this note is to strengthen one of the theorems on groups with projections proved by End in [1].

We recall the definitions of that paper. Let n be a natural number and let ω be $\{1,\dots,n\}$. Let G be a group. An *ω-structure* on G is a family of subgroups (G_α) $(\alpha\subset\omega)$ of G such that $G_\emptyset=\{0\}$ and

$$[x,y]\in G_{\alpha\cup\beta}\quad \text{for } \alpha,\beta\subset\omega, x\in G_\alpha,\ y\in G_\beta$$

($[x,y]$ denotes the commutator of x and y). The ω-structure (G_α) on G is said to be *universal* if the following condition holds: whenever $f_\alpha: G_\alpha\to H$ $(\alpha\subset\omega)$ is a family of homomorphisms such that

$$[f_\alpha x, f_\beta y]=f_{\alpha\cup\beta}[x,y]\quad \text{for } \alpha,\beta\subset\omega, x\in G_\alpha,\ y\in G_\beta,$$

then there is a unique homomorphism $f: G\to H$ such that

$$f_\alpha=f\,|\,G_\alpha\quad \text{for } \alpha\subset\omega.$$

For any ω-structure (G_α) on G and any total ordering $\le$ of the subsets of ω, we have a summation function

$$S^{\le}:\prod_{\alpha\subset\omega} G_\alpha\to G$$

given by the formula

$$S^{\le}(g_\alpha)=\sum_{\alpha\subset\omega} g_\alpha\quad \text{for } (g_\alpha)\in\prod_{\alpha\subset\omega} G_\alpha$$

(we write groups additively although they need not be commutative).

We shall prove the following result.

Theorem. *Let (G_α) be an ω-structure on a group G, and suppose that the summation function $S^{\le}:\prod_{\alpha\subset\omega} G_\alpha\to G$ is bijective for some total ordering $\le$ of the subsets of ω. Then the ω-structure (G_α) is universal.*

This is a generalization of Theorem C of [1], which needs a restriction on the ordering $\leq$.

The title of this paper is explained by Theorem A of [1], which gives a family of projections on a group with a universal ω-structure.

The theorem is proved in the same way as Theorem C of [1] is proved in [1], 12(c); we use the following lemma, which is analogous to the conjunction of the lemmas in [1], 12(a) and 12(b).

Lemma. *Assume the hypotheses of the theorem, and suppose that* $f_\alpha : G_\alpha \to H$ $(\alpha \subset \omega)$ *are homomorphisms such that*

$$[f_\alpha x, f_\beta y] = f_{\alpha\cup\beta}[x, y] \quad \text{for } \alpha, \beta \subset \omega, x \in G_\alpha, y \in G_\beta.$$

Let $(a_1, \ldots, a_k) \in G_{\alpha(1)} \times \cdots \times G_{\alpha(k)}$ *be any finite sequence of elements of the* G_α*'s. Then there is a sequence* $(b_1, \ldots, b_m) \in G_{\beta(1)} \times \cdots \times G_{\beta(m)}$ *such that*

$$\beta(1) < \beta(2) < \cdots < \beta(m),$$

$$a_1 + \cdots + a_k = b_1 + \cdots + b_m \quad \text{in } G,$$

$$f_{\alpha(1)}a_1 + \cdots + f_{\alpha(k)}a_k = f_{\beta(1)}b_1 + \cdots + f_{\beta(m)}b_m \quad \text{in } H.$$

The argument of [1], 12(c) shows that the lemma implies the theorem, so we shall only prove the lemma. We take the commutator $[x, y]$ to be $x + y - x - y$ for $x, y \in G$. Given a sequence $(a_1, \ldots, a_k) \in G_{\alpha(1)} \times \cdots \times G_{\alpha(k)}$ we shall perform the following operations on it: replace two consecutive elements $(a, b) \in G_\alpha \times G_\beta$ by

(i) $(a + b) \in G_\alpha$ if $\alpha = \beta$,

(ii) $([a, b] + b, a) \in G_\beta \times G_\alpha$ if $\alpha \subset \beta, \alpha \neq \beta$,

(iii) $(b, a + [-a, -b]) \in G_\beta \times G_\alpha$ if $\beta \subset \alpha, \beta \neq \alpha$,

(iv) $(b, [-b, a], a) \in G_\beta \times G_{\alpha\cup\beta} \times G_\alpha$ if $\alpha \subset \beta$ and $\beta \subset \alpha$ are both false.

(Note that $[a, b] \in G_{\alpha\cup\beta} = G_\beta$ in (ii); similarly $[-a, -b] \in G_\alpha$ in (iii).) Clearly these operations do not change the sums $\sum_i a_i$ in G and $\sum_i f_{\alpha(i)}a_i$ in H. It is therefore sufficient to show that the operations can be used to change the given sequence of indices $(\alpha(1), \ldots, \alpha(k))$ to a sequence $(\beta(1), \ldots, \beta(m))$ with $\beta(1) < \beta(2) < \cdots < \beta(m)$.

We proceed as follows. Write P_i for

$$\{\alpha \subset \omega : \alpha \text{ has cardinality } i\}$$

for $0 \leq i \leq n$. We transpose elements of P_1 with their neighbours by operations (ii)–(iv) until the elements of $P_0 \cup P_1$ occur in the order given by $\leq$; they may be repeated, and they may be mixed arbitrarily with elements of $P_2 \cup P_3 \cup \cdots \cup P_n$. Operation (iv) puts an extra element into the sequence as well as performing the desired transposition, but the extra element will be in $P_2 \cup P_3 \cup \cdots \cup P_n$, so it does not matter. After this we transpose the elements of P_2 with their neighbours in the same way until the elements of $P_0 \cup P_1 \cup P_2$ appear in the correct order. This may put in extra elements, but they will be in $P_3 \cup \cdots \cup P_n$, so they do not matter. We proceed in this way until all the elements appear in the correct order. The sequence then has the form $(\gamma(1), \ldots, \gamma(l))$ with $\gamma(1) \leq \gamma(2) \leq \cdots \leq \gamma(l)$. We now use operation (i) to eliminate the repeats, and so arrive at $(\beta(1), \ldots, \beta(m))$ with $\beta(1) < \beta(2) < \cdots < \beta(m)$ as desired.

This completes the proof.

Reference

[1] W. End, Groups with projections and applications to homotopy theory, J. Pure Applied Algebra 18 (1980) 111–123.